Differential equation using integrating factor 2

Solve the following differential equation

$$\left(\frac{y^2}{2} + 2ye^x\right) dx + (y + e^x)\, dy = 0$$

Solution

We see that it is not exact.

$$P'_y = y + 2e^x \quad \text{and} \quad Q'_x = e^x \quad \Rightarrow \quad P'_y \neq Q'_x$$

We calculate:

$$\frac{P'_y(x,y) - Q'_x(x,y)}{Q(x,y)} = \frac{(y+2e^x) - e^x}{y+e^x} = \frac{y+e^x}{y+e^x} = 1$$

We find the integrating factor:

$$\varphi(x) = e^{\int \frac{P'_y(x,y) - Q'_x(x,y)}{Q(x,y)} dx} = e^{\int 1 dx} = e^x$$

We multiply the entire equation by the integrating factor:

$$e^x \left(\frac{y^2}{2} + 2ye^x \right) dx + e^x \left(y + e^x \right) dy = 0$$

$$\left(\frac{y^2}{2} e^x + 2ye^{2x} \right) dx + \left(ye^x + e^{2x} \right) dy = 0$$

We check that this equation is exact.

$$P'_y = ye^x + 2e^{2x} \quad \text{and} \quad Q'_x = ye^x + 2e^{2x} \quad \Rightarrow \quad P'_y = Q'_x$$

We solve the exact differential equation.
We look for $U(x,y)$:

$$U(x,y) = \int Q(x,y) dy = \int \left(ye^x + e^{2x} \right) dy = \frac{y^2}{2} e^x + ye^{2x} + C(x)$$

We derive with respect to x.

$$U'_x = \frac{y^2}{2} e^x + 2ye^{2x} + C'(x)$$

We find $P(x,y)$ and then equate with U'_x

$$\int ye^x + 2e^{2x} dy = \frac{y^2 e^x}{2} + 2ye^{2x} = \frac{y^2}{2} e^x + 2ye^{2x} + C'(x)$$

$$C'(x) = 0 \quad \Rightarrow \quad C(x) = C$$

Substituting, we get the solution:

$$\frac{y^2}{2} e^x + ye^{2x} + C = 0$$